# Snakes on a Plain

Use pi to find the length of rivers that wind their way across the plains.

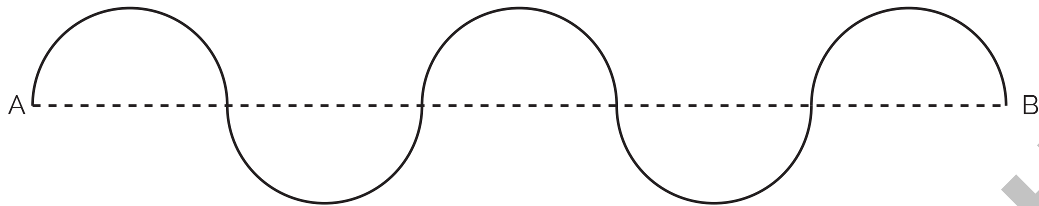



Figure 1: The Five-Snake River

## Introduction

In my dream last night, I was trapped in an aluminum cylinder that was hovering precariously 5 mi above the Earth's surface. The only thing I had to eat was a tiny bag of five peanuts. Every couple of minutes the cylinder would shake unpredictably, thus reminding me of my delicate position and the terrifying demise that would result if the cylinder were to stop hovering. Oh, wait, that wasn't my dream; it was my flight back from Wichita.

I'm not a big fan of flying. I understand the physics of flight, and I am aware that I am much more likely to die in a bathtub accident or at the hands of a shark than I am to die in a plane crash, which is amazing because sharks don't even have hands. (I think the difference in probability is even more profound if the shark is actually IN your bathtub.)

Even though I know all this, I am still unnerved by planes. The whole thing seems unnatural to me. If God had intended for us to fly, he or she would have a left us a note letting us know. "Dear, humans, you can fly. I am leaving you this note to let you know because otherwise it might not be so apparent to you… what with the whole gravity thing and all… plus your lack of wings. Also, sorry about the mosquitoes. I don't know what I was thinking."

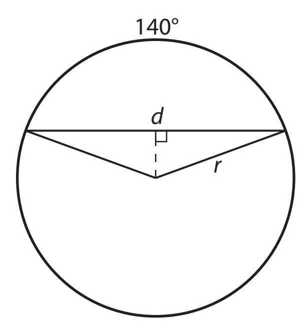
So, because I don't like flying, I look for ways to distract myself. One of my favorites is to look out the window and try to match geographic features with what I can see in the atlas I brought along for this purpose. While doing this on my flight from Wichita, I became fascinated by how the rivers always get from point A to point B in a back-and-forth snaking manner. If we saw a man doing this, we would stare and perhaps contact the

### Fun River Facts

- ✓ There are 3,500,000 mi of rivers in the United States. Here's the surprise: You co-own a large percentage of these rivers because they are public domain up to their high-water marks.
- ✓ There are 18 countries that don't have a single river.
- ✓ Russia is the country with the most rivers. They have about 100,000!

### Itty-Bitty Hint

- ✓ In Challenge 6, you'll need to relate the length of the arc to the length of the chord. To do that, you'll first need to relate the length of the chord to the radius of the circle of which the arc is a part.





authorities, but with rivers we understand that it is just a natural consequence of how they are formed.

## Challenge 1

1. Figure 1 on the previous page shows a meandering river that has been modeled by five semicircles. The teacher will give you a value for *AB*. Use this straight-line distance to find the length of the Five-Snake River.
2. Hmmm, I wondered to myself on the Wichita flight, how will this river length change if there are more congruent semi-circles? Using the same value for *AB*, calculate the length of the Ten-Snake River and the Two-Snake River.
3. Have the teacher initial your work.

## Challenge 2

4. Algebra can be a powerful tool for analyzing problems like this one. Let *n* = the number of semi-circles, *AB* = the straight-line distance from *A* to *B*, and *L* = the total length of the river. Derive a formula for calculating *L* from the other variables.
5. If you did this correctly, you should get a result that confirms what you discovered in the first challenge.
6. Have your teacher initial your work.

## Challenge 3

Let's turn up the level of fun. Suppose the river is like the one below. It is made up of five non-congruent semicircles with diameters as shown.

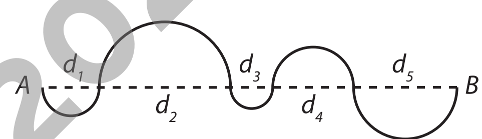



Figure 2: The Wiggly River

7. Make a diagram of your Wiggly River. Label your five diameters with values that add up to *AB*.
8. Calculate the length of the river, and compare it to your answers from Challenge 1.
9. Have your teacher initial your work.

## Challenge 4

It's now time for the grand finale of this part of the project. Imagine splitting the river up into *five* non-congruent semicircles. Let the diameters of the semicircles be $d_1, d_2, \ldots, d_n$. Note that these diameters add up to the length of the river as shown by this equation:

$$D = d_1 + d_2 + d_3 + \ldots + d_n$$

10. Derive a formula for calculating the total length of the river, *L*. Simplify this formula as much as you can. The distributive property is your friend.
11. How does this compare to your previous results using congruent semicircles?
12. Explain whether your discovery could or could not use 100 non-congruent circles.
13. Have the teacher initial your work.

## Challenge 5

14. If you had derived the formula in Challenge 4 from the beginning, you would not have needed to do all the other work leading up to it. This shows how algebra can serve as a language that reveals the truth about a phenomenon. Pretty cool, huh? Now you have the chance to get in on the ground floor. Use algebra to show how the circumferences of the smaller circles relate to the circumference of the larger circle.

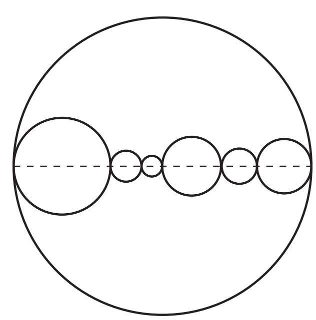
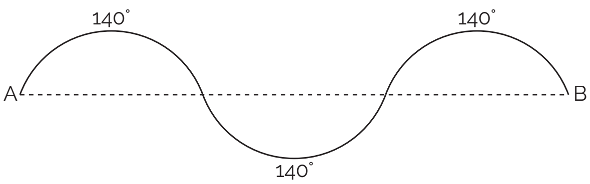


Figure 3: Circles in Circles

## Challenge 6

Are you dancing on the desk because of the coolness of your discovery? Have you considered asking the teacher to reward you with… your own plane?

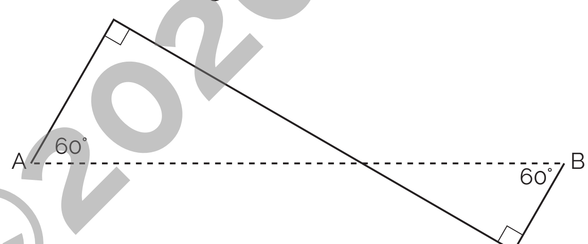
Hold on because we're just getting started. We've been exploring snaking rivers made of semicircles. What if the rivers were made of other shapes?




15. The diagram above shows a river made of three congruent 140° arcs. Choose a value for *AB*, and calculate the length of this river.
16. Sketch an arc-based river using some other number of 140° arcs. Calculate the length of this river.
17. Now derive a generic formula for the length of a river based on *n* 140° arcs of different sizes. (You did this for semicircles in Challenge 4.)
18. You used arcs with the same measure. Explain why you would or would not get these same results if you use arcs with different measures.
19. Have the teacher initial your work.

## Challenge 7

Let's see what happens when we use rivers that are not based on circles at all. Here's a river made of 30°–60°–90° triangles.




20. This time let's jump right to the most generic case possible. Derive a formula for calculating the length of a river made from *n* 30°–60°–90° triangles.
21. Explain why this did or did not turn out just like the results for the *n*-semicircle case.

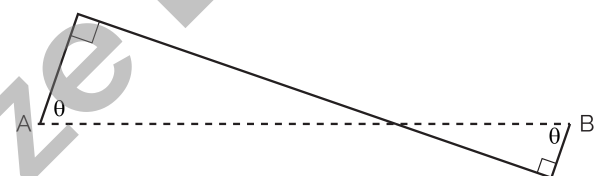
Let's represent the formulas you have been deriving with one generic formula:

$$L = k \cdot AB$$

22. In this formula, *k* represents a constant. Determine and list the values of *k* for the semicircular river, the 140°-arc river, and the 30°–60°–90° triangle river.
23. Have the teacher initial your work.

## Challenge 8

Still want more? How about a river made of right triangles in which one of the non-right angles measures θ? (Pick a value.) If this gives you the same results that you've been getting, you'll have the most generic result possible for right triangles.




24. Derive a formula using trigonometry that you can use to find the length of an *n*-triangle river.
25. Have the teacher initial your work.

## Challenge 9

26. Represent your results from Challenge 8 using a constant, *k* (like you did in 21). Write an expression for *k* in terms of θ.
27. Use your graphing calculator to find the value of θ for which the river is longest. What is the largest possible value for *k*?
28. Have the teacher initial your work.

## Challenge 10

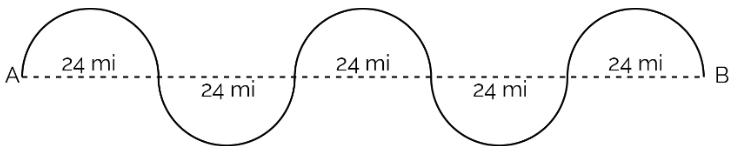
Explain how you could avoid this whole project and derive a single result if you view the rivers as being made of similar figures.

# Snakes on a Plain

**Teacher Stuff:** Solutions with Teaching Tips

## CHALLENGE 1

**Solution 1** The answer presented here is for $AB$ = 120 mi. Students will have different values for $AB$.




Divide $AB$ into five congruent segments. Then find the length of one semicircle.

$$\text{Length of one semicircle} = \frac{24\pi}{2} = 37.699 \text{ miles} \qquad (1\text{-}1)$$

Multiply by the number of semicircles.

$$\text{Total length of river} = 37.699 \cdot 5 = 188.5 \text{ mi} \qquad (1\text{-}2)$$

**Solution 2** As students repeat this calculation for different numbers of semicircles, they discover the same answer.

### Teaching Tip #1

You'll need to give each student or team of students a different value for $AB$. This has two advantages. First, it obviously keeps students from just sharing answers. More importantly, later when you do discuss answers as a class, students will see that the results are independent of any specific river length.

By the way, you can also give different groups of students rivers with numbers of semicircles other than 2 and 10. Some of your students will experiment on their own.

## CHALLENGE 2

**Solution 4** Students should follow the pattern of the examples they have completed.

Divide $AB$ by the number of semicircles.

$$\text{Length of one segment} = \frac{AB}{n} \qquad (2\text{-}1)$$

Find the length of one semicircle.

$$\text{Length of one semicircle} = \frac{\pi AB}{2n} \qquad (2\text{-}2)$$

Multiply by the number of semicircles.

$$\text{Total length of } \frac{\pi AB}{2n} \cdot n = \frac{\pi AB}{2} \qquad (2\text{-}3)$$

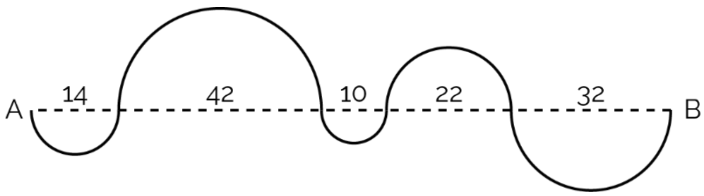
### Teaching Tip #2

This step presents a very important teachable moment. When you check in with students before initialing their work, ask them if they noticed what disappeared in the final expression. Then ask them what that says about which variables affect or don't affect the length of the river. Their understanding of this is essential to the rest of the project.

**Solution 5** When you initial student work, they should be able to tell you that they notice that $n$ is not in the expression. As they already discovered, the length of the river is independent of the number of semicircles.

## CHALLENGE 3

As before, students' rivers and the way they split them up will differ. The procedure is the same.




**Solution 7** $14+42+10+22+32=120 \text{ mi}$ (3-1)

**Solution 8** Answers will vary, but all students should choose segment lengths that add up to $AB$.

$$\frac{\pi 14}{2}+\frac{\pi 42}{2}+\frac{\pi 10}{2}+\frac{\pi 22}{2}+\frac{\pi 32}{2}=188.5 \quad (3\text{-}2)$$

### Teaching Tip #3

Students are very likely to approach this in a more haphazard fashion than what is shown here. When you check in with them to sign, you can help them see that this is a way to organize the thinking they have already done. This will help them with the next step.

## CHALLENGE 4

Students have already done the thinking for this step. (See Teaching Note #3.) Now it's a matter of doing it with variables to represent the diameters.

**Solution 10** Add the length of $n$ semicircles.

$$d_1+d_2+d_3+\ldots\ +d_n \quad (4\text{-}1)$$

Replace the sum of the diameters with $D$.

$$L=\frac{\pi D}{2} \quad (4\text{-}2)$$

### Teaching Tip #4

This is the coolest part of the whole project. If your students are like ours, they'll be pretty excited when they make this discovery.

Students will need varying amounts of hinting to get here, but the more you can let them arrive at this result on their own, the stronger they'll be for having done so.

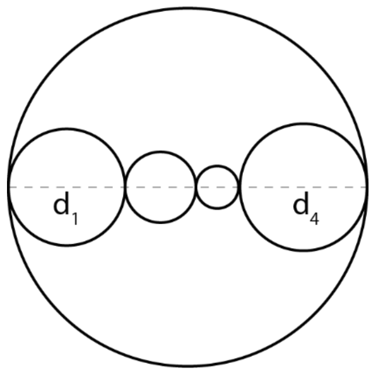
**Solution 11** Students should be able to tell you that this is the same result that they got from the same three challenges.

**Solution 12** They should also be able to reason that this result will occur for any number of segments that add up to $AB$.

## CHALLENGE 5

**Solution 14** This problem is structured more loosely on purpose. Students have an opportunity to analyze this new situation using some of the patterns and thinking from the first four challenges. Some students may solve the problem first with congruent circles before moving to an example with circles that have different diameters. Others might be able to move to the highest level of generalization right away.

This generalized solution for four circles is shown here.




Represent the circumference of each circle, and then add them.

Sum of circumferences = $\pi d_1 + \pi d_2 + \pi d_3 + \pi d_4$ (5-1)

Factor out $\pi$. Let $D$ = diameter of the big circle.

$\pi(d_1 + d_2 + d_3 + d_4) = \pi D$ = circumference of big circle. (5-2)

This challenge is a great opportunity to talk about how to develop a clearly written solution.

### Teaching Tip #5

Giving students an opportunity to think independently through an analysis helps them develop a skill that employers desperately want. In this case, students have had the advantage of being led first through a similar analysis, but that does not always need to be the case.

### Teaching Tip #6

The conversations you'll have with your students as they create differing solutions to this problem are priceless—for you as well as for them. The more you create this type of opportunity for students, the less you'll want to be lecturing from the front of the room.

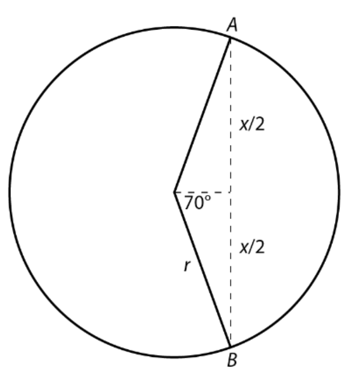
## CHALLENGE 6

**Solution 15** Let's calculate the length of a single arc, as shown by $\widehat{AB}$.

$$r = \frac{x}{2\sin 70°} = 0.5321x$$

$$\text{Arc length} = \frac{140}{360} \cdot 2\pi r = 1.30x$$

(6.1)




### Teaching Tip #7

You can decide whether or not the rest of the challenges are right for your students. The expectation is that students will be more independent in making connections and generalizing and explaining results.

$$\text{total river length} = 3 \bullet \frac{120}{3} \bullet 1.30 = 156 \text{ mi} \qquad (6\text{-}2)$$

**Solution 16** Students should get the same result no matter how many arcs they use. Some students may want to skip this step because they already see that all we have changed is the fraction by which we multiply the circumference. In our opinion, this is okay as long as they can provide a clear explanation that demonstrates that they understand the principle involved.

**Solution 17** Here is the generalized result:

$$1.30d_1 + 1.30d_2 + \ldots + 1.30d_n = 1.30\left(d_1 + d_2 + \ldots + d_n\right) \qquad (6.2)$$

Finally, $L = 1.30D$, which should seem familiar. (6.3)

**Solution 18** The 1.30 that was factored out is directly related to the arc measure, so changing this arc measure for each section of the river will prohibit you from using the distributive property and getting tidy results.

## Challenge 7

**Solution 20** Let $d$ represent the length of one segment when $\overline{AB}$ is split up. The lengths of the two sides of the triangle in that section are then $\frac{1}{2}d$ and $\frac{\sqrt{3}}{2}d$. Combine these to represent the length of one section of the river for one triangle: $d\left(\frac{1+\sqrt{3}}{2}\right)$. If you add expressions for each of the river sections, you'll get:

$$L = \left(\frac{1+\sqrt{3}}{2}\right)\left(d_1 + d_2 + d_3 + \ldots + d_n\right) = \left(\frac{1+\sqrt{3}}{2}\right)D \qquad (7.1)$$

**Teaching Tip #8**
There are of course other possibilities to explore with specific types of triangles. Maybe some groups can use these triangles while others use isosceles triangles or 3:4:5 triangles. Students have analyzed the basic concepts enough by this point to be able to explore and play.

**Solution 21** For all of our examples, the length of one section of the river is found by multiplying the straight-line length of that section by some constant. As long as the river is made up of similar shapes, each section length will be multiplied by this same constant, which can later be factored out of the sum of the sections.

**Solution 21** semicircular river: $k$ = 1.57; 140°-arc river: $k$ = 1.22; 30°–60°–90° river: $k$ = 1.37.

## CHALLENGE 8

The formula development for this triangle is the same as for the 30-60-90° triangle. The two sides will be $d\sin\theta$ and $d\cos\theta$. One section can be represented by $d\left(\sin\theta+\cos\theta\right)$.

**Solution 24** The result is $L=\left(\sin\theta+\cos\theta\right)D$. (8-1)

## CHALLENGE 9

**Solution 26** $k=\sin\theta+\cos\theta$ (9-1)

**Solution 27** Graphing this function shows that the largest value for *k* turns out to occur when $\theta=45°$. At that angle, *k* = 1.414. (Feel free to find this same result using calculus… just for fun.)

### Teaching Tip #9

Are you and your students still hungry for more? Try designing rivers that result in specific values for *k*. How about *k* = 1.5 or *k* = 2? You already know that triangular rivers and rivers based on arcs won't work. Why not? What other shapes could work? There is room for creativity in the math classroom. Have fun!

## CHALLENGE 10

It's weird to think that after all this analysis the idea that yields such cool results is simple: similar figures. In similar figures, a given distance will be equal to some other distance multiplied by a constant. Continuing with the variables used throughout this project, we can write:

$$L=kd_1+kd_2+\ldots+kd_n=kD \qquad (10.1)$$

That's really all there is to it. It will be interesting to see if any of your students think of this generalization before they get to this last challenge.